# Convexity, Helly's Theorem, and Approval Voting

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**Abstract**

We investigate and prove several theorems about the intersections of multiple convex sets, focusing on Helly's Theorem. We then look at some contemporary research on an application to approval voting, using a model based on Helly's Theorem in various dimensions. We conclude with some open questions and possible directions for future research.

## 1 Introduction

A common way of describing someone's political preferences is through a political compass, such as one found at `https://www.politicalcompass.org/` [6]. Pictured below in Figure 1 are 31 presidential 2020 candidates, plotted with their positions on an economic scale as well as a social scale.

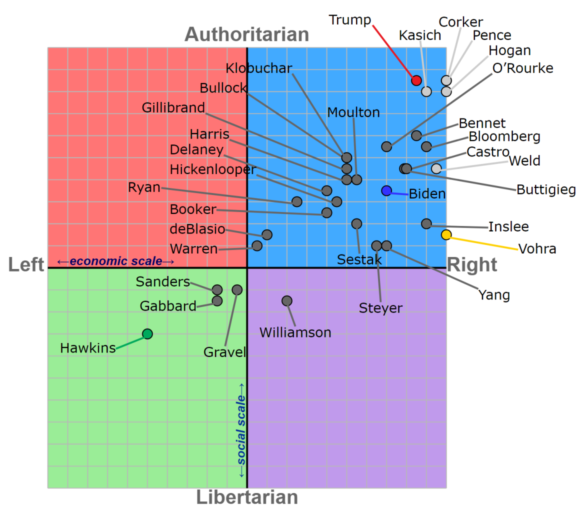


Figure 1: 2020 US presidential candidates on a political compass, [6].

If each voter in the United States wanted to vote for a candidate that precisely equals their own beliefs, they would be disappointed, since the majority of possible spaces on the grid are open (for example the origin). If we, instead of just considering each singular point, represent each candidate by a disk of diameter 5, we get the collection of disks in Figure 2.

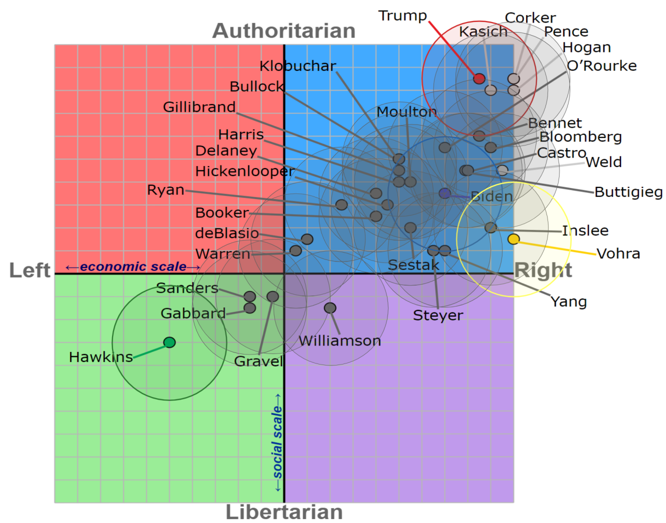


Figure 2: Each candidate is appealing to people who vote in a disk surrounding the candidate's defined platform, [6].

How do we interpret this? What is the behavior of the complex web of overlapping disks in the blue quadrant? What are the effects of changing the radius of the disks? What if the disks were replaced by squares?

In order to investigate, we turn to Helly's Theorem. Helly's Theorem states that if we have a family of convex sets in some dimension $d$, the number of sets is more than the number of dimensions, and every collection of $d + 1$ of those sets has a nonempty intersection, then the intersection of all the sets is nonempty. While our specific scenario of presidential candidates does not satisfy this hypothesis, the connection between this theorem and our questions is apparent.

Eduard Helly was born in Vienna in 1884, and eventually got his Ph.D. in the same city in 1907. Helly made many mathematical discoveries across the fields of geometry and analysis, with perhaps his most well-known result being the theorem upon which this paper focuses. Helly discovered his theorem in 1913, but he did not formally publish his findings for 10 years. His correspondence with Johann Radon, in the meantime, led to two proofs being published in 1921 and 1922 (the first by Radon himself, and the second by Dénes König).

Helly's Theorem is related to Radon's Theorem and Carathéordory's Theorem, as the three all provide insights into the nature of certain convex sets. While we won't discuss Carathéodory's Theorem in this paper, we will make clear the logic of one proof of Helly's Theorem when we progress to Section 3, where we investigate Johann Radon's proof from 1921.

After appreciating the nuances of Helly's Theorem, in Section 4 we will look at some of its applications in the 21$^{\text{st}}$ century. Through the work of several contemporary authors, the fundemental concepts of Helly have been brought to life in the context of the approval voting system.

# 2 Understanding Convexity

Having established our motivation, we shall dive right into defining the fundamental concepts that this topic rests on. We first must gain an understanding of convexity.

A set $P$ is "convex" if, for any two points $x, y \in P$, the line segment $\overline{xy}$ between them is totally contained in $P$. In Figure 3, we see that circles, squares, and regular heptadecagons are convex, but annuli, stars, and crescents are not.

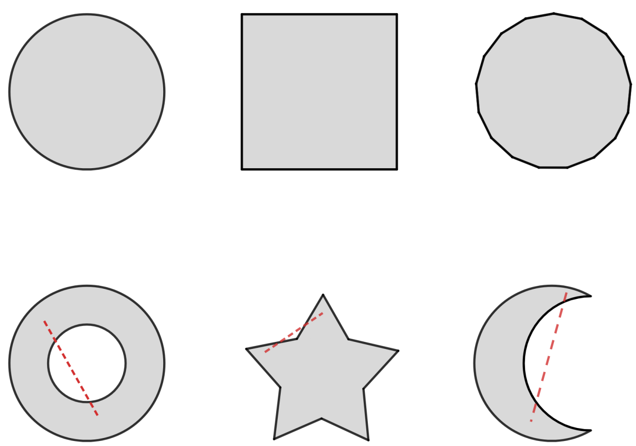

Figure 3: The three sets on top are convex sets in two dimensions. Note the red dashed lines in the bottom three non-convex sets: they show the existence of an $\overline{xy}$ that is not fully contained in the set.

We formally define convexity in the following algebraic sense, from [7]:

**Definition 1.** *A subset $P$ of $\mathbb{R}^n$ is said to be* **convex** *if*

$$(1-\lambda)x + \lambda y \in P \tag{1}$$

*whenever $x \in P, y \in P$, and $0 < \lambda < 1$.*

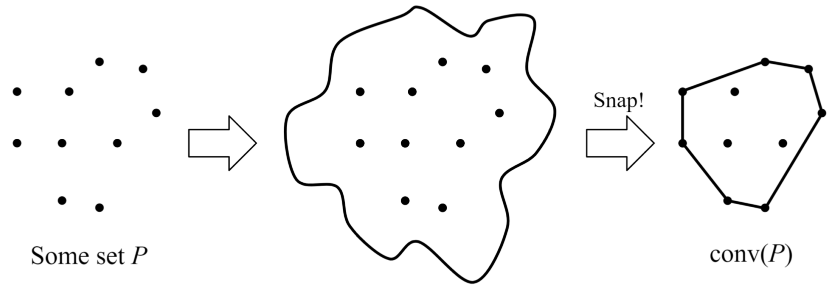


Figure 4: The "rubber-band" method of obtaining conv(P)

By taking $\lambda$ as our parameter, we see that the value of (1) changes from $x$ to $y$ as $\lambda$ increases, staying between the two exterior points. Keep this concept in mind as we continue defining the different tools of convexity.

The "convex hull" of a set $P$ is the intersection of all convex sets containing $P$; equivalently, it is the unique convex set containing $P$ that is a subset of all other convex sets containing $P$. We can get an intuitive understanding of this by imagining a rubber band (Figure 4) stretched around the points of $P$, then letting go, allowing only the "outside" points of $P$ to define the shape of the hull.

To formalize the definition of a convex hull, we turn to an analogue of a linear combination of elements. If we take a linear combination of $m$ points $x_i \in \mathbb{R}^n$, we get an expression of the form

$$\sum_{i=1}^{m} \alpha_i x_i = \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_m x_m,$$

where $\alpha_i \in \mathbb{R}$. Placing additional restrictions on this general form gives us a tool useful for describing convex sets.

**Definition 2.** *A* **convex combination** *of points $x_i$ for some index set $i \in \mathcal{I}$ is a linear combination of the form*

$$x = \sum_{i \in \mathcal{I}} \alpha_i x_i = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + ...$$

*where $\alpha_i \geq 0$ and $\sum_{i \in \mathcal{I}} \alpha_i = 1$.*

Definition 2 of a convex set utilized the convex combination of two points. The definition of "convex combination" is crucial to the study of convex sets, as we can now construct convex sets out of any set we like in $\mathbb{R}^d$ using it.

**Definition 3.** *Let $P$ be a subset of $\mathbb{R}^d$. The* **convex hull** *of a set $P$, denoted* **conv(*P*)** *is defined as*

$$\text{conv}(P) = \{a \in \mathbb{R}^n \mid a \text{ is a convex combination of points in } P\}.$$

This definition is equivalent to the geometric understandings stated above.

# 3 Helly's Theorem

Now that we know what convex sets are, we can start investigating the behavior of their intersections. When we want to know about a large number of convex sets, we will ultimately be lead to Helly's Theorem.

**Theorem 1. (Helly's Theorem.)** *Let $X_1, X_2, X_3, ..., X_n$ be convex subsets of $\mathbb{R}^d$, where $n, d \in \mathbb{N}$, and $n > d$. If the intersection of every collection of $d+1$ of these sets is nonempty, then the simultaneous intersection of all $n$ sets is nonempty.*

In other words, the theorem first requires that we have a collection of convex sets in dimension $d$, such that the number of sets is larger than the dimension. If every time we pick a collection of $d+1$ of those sets and they all overlap somewhere, then we know that *all* $n$ of the sets overlap.

For example, suppose we have a collection of 19 convex sets in the Cartesian plane, and we know that every trio of them has a nonempty intersection. Then by Helly's Theorem, we know that the simultaneous intersection of all 19 sets is nonempty! We are able to draw a conclusion of 19 sets from the behavior of trios of sets. We require every *trio* to have a nonempty intersection because the space we are in is 2-space, so $d = 2$, and Helly's Theorem requires all collections of $d + 1 = 3$ sets to coincide.

Helly's Theorem requires some work to prove, so before we prove it in Section 3.2, we start by proving some neccessary propositions.

## 3.1 Propositions Leading Up to Helly's Theorem

We first investigate the relationships between two individual convex sets. A natural question to ask is what the intersection between them and the union of them will look like. We find first that the union of two convex sets is not necessarily convex. Consider $[0, 1], [2, 3] \in \mathbb{R}$: these sets are both convex, but their union is not. On the other hand, for the intersection of these two intervals, we get the empty set $\emptyset$, which is trivially convex (by inability to contradict). Extending this finding, we show:

**Theorem 2.** *If $A, B \in \mathbb{R}^n$ are convex sets, then $A \cap B$ is convex.*

*Proof.* Let $A, B \in \mathbb{R}^n$ be convex sets. Let $x, y \in A \cap B$, and consider $(1-\lambda)x + \lambda y$, with $0 < \lambda < 1$. Since $x, y \in A \cap B$, this implies that $x, y \in A$. Since $A$ is a convex set, then for $0 < \lambda < 1$, we have $(1-\lambda)x + \lambda y \in A$. Using a similar argument with respect to the